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Operator Equations and Nonlinear Eigenparameter Problems

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Theorems concerning the existence and the approximation of roots of operator equations in an abstract space are established and then applied, advantageously, to extend the theory of nonlinear eigenparameter problems.

1. INTRODUCTION

For a given one parameter family of operators $L(\lambda)$ we are interested in determining the values of λ for which $L(\lambda)$ does not have a bounded inverse which is an endomophism. To emphasize that we are dealing in cases in which the dependence of L on λ is not linear we call this a nonlinear eigenparameter problem. The theory of nonlinear eigenparameter problems, rich with applications, enjoys a vast literature [1]–[80] and a variety of techniques have been brought to bear on the subject. Krein and Langer [42, 53] (cf. also [14]) attack the problem via the theory of operator equations. Their results apply to cases in which $L(\lambda)$ is a quadratic polynomial in λ having self-adjoint operator coefficients.

In this paper we apply the contraction mapping principle to construct roots of operator equations. The roots are then used to gain information for expressions $L(\lambda)$ depending analytically on λ. We prove a theorem which is similar to a theorem in [42] but which applies more generally in a Banach space. An important advantage, gained from the contraction mapping principle, is that approximate solution with error bounds are provided for.

The use of the contraction mapping principle in dealing with polynomial operators has been observed by L. B. Rall and Patricia M. Prenter (cf. the survey article by Prenter in [82]).

Operator equations have been considered briefly by Lancaster [47] and Lovass-Nagy and Powers [81].

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2. Roots of Operator Equations

Let $(\mathscr{X}, +, |\cdot|)$ denote a set $\mathscr{X}$ whose elements form a group under $+$ and $\mathscr{X}$ is a complete metric space with respect to the metric $d(x, y) = |x - y|$. One might think of the complete normed group $(\mathscr{X}, +, |\cdot|)$ as an F-space where scalar multiplication is disregarded. The set of mappings $A : \mathscr{X} \to \mathscr{X}$ such that (1) $A(x + y) = Ax + Ay$, $x, y \in \mathscr{X}$ and (2) $|A| = \sup_{|x|>0} |Ax|/|x| < \infty$ form the induced complete normed ring $(\mathscr{R}, +, |\cdot|)$. An example of such a space $\mathscr{X}$ for which partial differential operators with smooth coefficients $\in \mathscr{R}$ is $\mathscr{C}(\Omega)$ (cf. [83, p. 52]).

For given $A_{ij} \in \mathscr{R}$, $i = 0, 1,\ldots, 0 \leqslant j \leqslant i$, we wish to calculate $Z \in \mathscr{R}$ such that

$$Z = \sum_{i=0}^{\infty} \sum_{j=0}^{i} Z^j \quad A_{ij} Z^{i-j} \equiv F(Z) \tag{2.1}$$

THEOREM 2.1. *Let $f(b) < b$ where b is the positive number (assumed to exist) for which the derivative of the function*

$$f(\alpha) = \sum_{i=0}^{\infty} \left[\sum_{j=0}^{i} |A_{ij}| \right] \alpha^i$$

takes on the value one. Then:

(1) *there is a unique nonnegative number a such that $a < b$ and $f(a) = a$;*

(2) *for every α, $a \leqslant \alpha < b$, F is a contraction of the sphere of radius α;*

(3) *for any choice of Z_0 in the sphere of radius b the sequence of iterates $Z_{n+1} = F(Z_n)$ converges in $\mathscr{R}$ to a unique root $Z_\infty \in \mathscr{R}$ of (2.1) and $|Z_\infty| \leqslant a$;*

(4) *If $|Z_0| \leqslant \alpha$, where $a \leqslant \alpha < b$, then*

$$|Z_n - Z_k| \leqslant \frac{(f'(\alpha))^n}{1 - f'(\alpha)} |Z_1 - Z_0|, \; n \leqslant k \leqslant \infty; \; 0 \leqslant f'(\alpha) < 1; \tag{2.3}$$

(5) *if A_{00} is compact,* i.e., *if A_{00} takes bounded subsets of $\mathscr{X}$ into sequentially compact subsets of $\mathscr{X}$,then Z_∞ is compact.*

(6) *if scalar multiplication is defined in $\mathscr{X}$ and the coefficients are linear,* i.e., *$A_{ij}(\beta x) = \beta A_{ij} x$, $x \in \mathscr{X}$, $i \geqslant 0$, $0 \geqslant j \leqslant i$, and for all scalars β, then Z_∞ is linear.*

Remark. It will be seen from the contraction mapping principle as it is stated in [11] (p. 150) and from the proof of the theorem that it is not essential to assume that $|\cdot|$ is definite, i.e., $|x| > 0$ if $x \neq 0$. Although the removal of this condition makes for a clumsier theorem.

Proof of Theorem 2.1. For (1), $f(\alpha)$ is increasing with $f(0) \geqslant 0$ and $f(b) < b$ hence $f(\alpha)$ has a unique fixed point in $[0, b]$.

For (2), we observe that for $a \leqslant \alpha < b$ and $|X| \leqslant \alpha$, $|Y| \leqslant \alpha$,

$$\text{(a)}\quad |F(x)| \leqslant f(\alpha) \leqslant \alpha, \qquad \text{(b)}\quad |F(x) - F(y)| \leqslant f'(\alpha)|x - y|. \tag{2.4}$$

The first statement is obvious from the definition of f, the choice of a and b, and the norm property $|XY| \leqslant |X||Y|$. For (2.4b) we write

$$|F(X) - F(Y)| = \left| \sum_{i=1}^{\infty} \sum_{j=0}^{i} \left[\sum_{m=0}^{j-1} X^{j-m-1}(X - Y)Y^m A_{ij} X^{i-j} \right.\right.$$
$$\left.\left. + \sum_{m=j}^{i-1} Y_j A_{ij} X^{i-m-1}(X - Y)Y^{m-j} \right]\right|$$

(the index m takes only nonnegative integer values) and use the triangle inequality.

For (3), we have, from the contraction mapping principle, that the sequence of iterations $\{Z_n\}$ converges to a root Z_∞, unique in the sphere of radius α and $|Z_\infty| \leqslant \alpha$. This being satisfied for all α in $[a, b)$ we must have $|Z_\infty| \leqslant a$. If $|Z_0| = b$ then, because $f(b) < b$, $|Z_1| < b$ and the sequence of iterates Z_1, Z_2,..., lies in one of the spheres of radius α, $a \leqslant \alpha < b$, and hence converge to the root Z_∞, which is necessarily unique in the sphere of radius b.

The approximation (2.3) is seen from the proof of the contraction mapping principle.

Conclusions (5) and (6) follow from taking $Z_0 = A_{00}$ and noting that the uniform limit of compact (linear) maps is compact (linear).

As an example of the applicability of Theorem 1.1 consider the quadratic equations of the type

$$\text{(a)}\quad \hat{A}Z^2 + Z + \hat{C} = 0, \qquad \text{(b)}\quad Z^2\hat{A} + Z + \hat{C} = 0, \tag{2.5}$$

where $|\hat{A}| > 0$. Here, in the notation of the theorem,

$$a = (1 - d)/(2|\hat{A}|), \qquad b = (2|\hat{A}|)^{-1}, \tag{2.6}$$

and the appicability condition $f(b) < b$ is

$$d = (1 - 4|\hat{A}||\hat{C}|)^{1/2} > 0. \tag{2.7}$$

3. A Pair of Complete Roots

We consider the quadratic equation

$$AZ^2 + BZ + C = 0, \tag{3.1}$$

where A, B, C, $B^{-1} \in \mathscr{R}$ and if we set

$$\hat{A} = B^{-1}A, \hat{C} = B^{-1}C \tag{3.2}$$

then condition (2.7) is satisfied. The application of Theorem 2.1 to Eqs. (2.5) yields roots Z_1, X_1 such that

$$AZ_1^2 + BZ_1 + C = 0, \qquad | Z_1 | \leqslant a; \tag{3.3}$$

$$X_1^2\hat{A} + X_1 + \hat{C} = 0, \qquad | X_1 | \leqslant a. \tag{3.4}$$

Assuming now that $A^{-1} \in \mathscr{R}$ (cf. Proposition 5.2) we find a second root of (3.1)

$$Z_2 = -\hat{A}^{-1} - \hat{A}^{-1}X_1\hat{A} \tag{3.5}$$

satisfying

$$AZ_2^2 + BZ_2 + C = 0, \qquad | Z_2 | \leqslant | \hat{A}^{-1} | (1 + | \hat{A} | a). \tag{3.6}$$

A pair of roots Z_1, Z_2 are said to be complete if $Z_1 - Z_2$ has a bounded inverse. For the roots under consideration, we have $\hat{A}(Z_1 - Z_2) = I - D$ where I is the identity map and

$$D = -X_1\hat{A} - \hat{A}Z_1, \quad | D | < 1 - d < 1. \tag{3.7}$$

Thus the pair Z_1, Z_2 is complete and

$$(Z_1 - Z_2)^{-1} = \sum_{n=0}^{\infty} D^n\hat{A}, |(Z_1 - Z_2)^{-1} | \leqslant | \hat{A} |/d. \tag{3.8}$$

Furthermore, the root Z_2 has a bounded inverse, for $-\hat{A}Z_2 = I + X_1\hat{A}$ and $| X_1\hat{A} | \leqslant (1 - d)/2 < 1$. This gives

$$Z_2^{-1} = -\sum_{n=0}^{\infty} (-X_1\hat{A})^n\hat{A}, | Z_2^{-1} | \leqslant b_*^{-1}, b_* = (1 + d)b. \tag{3.9}$$

Remark. In the above discussion the assumption that A have a bounded inverse in $\mathscr{R}$ is not essential. If we require only that $\hat{A}$ have a right inverse in $\mathscr{R}$, i.e., there is a $\hat{A}^r$ in such that

$$\hat{A}\hat{A}^r x = x, \qquad x \in X, \tag{3.10}$$

then the second root

$$Z_2 = -\hat{A}^r - \hat{A}^r X_1 \hat{A} \tag{3.11}$$

is still defined but Z_1, Z_2 may not be complete. However, $\hat{A}(Z_1 - Z_2)$ has a bounded inverse

$$[\hat{A}(Z_1 - Z_2)]^{-1} = \sum_{n=0}^{\infty} D^n. \tag{3.12}$$

As for the invertability of the root Z_1, we observe that, even when $\hat{A}$ is not invertable, $I + \hat{A}Z_1$ has an inverse

$$(I + \hat{A}Z_1)^{-1} = \sum_{n=0}^{\infty} (-1)^n(\hat{A}Z_1)^n, \ | (I + \hat{A}Z_1)^{-1} | \leqslant 2(1 + d)^{-1}, \tag{3.13}$$

and hence

$$-Z_1 = (I + \hat{A}Z_1)^{-1}\hat{C} \tag{3.14}$$

has an inverse in $\mathscr{R}$ if and only if C has an inverse in $\mathscr{R}$.

It is easy to construct examples for which an equation of the type (2.5) has no roots. Let A, B, and C be $n \times n$ matrices for which A is the identity matrix and B commutes with C. Further, choose B and C so that the minimal polynomial of the matrix $B^2 - 4C$ has a double root at the origin (cf. [81]).

It is also easy to see that the quadratic formula applies in obtaining roots of equations of the form $Z^2 + BZ + C = 0$ if and only if B commutes with C and $B^2 - 4C$ has a square root.

4. Nonlinear Eigenparameter Problems

Let A_i, $i = 0, 1, \ldots$ be bounded linear operators in a (complex) Banach space $(\mathscr{B}, \| \cdot \|)$ such that the analytic function

$$g(\lambda) = \sum_{i=0}^{\infty} \| A_i \| \lambda^i \tag{4.1}$$

is entire. The spectrum, $\sigma(L)$, of the expression

$$L(\lambda) = \sum_{i=0}^{\infty} \lambda^i A_i, \tag{4.2}$$

is the set of complex values λ for which $L(\lambda)$ does not have an inverse in $\mathscr{R}$, the ring of bounded linear operators on $\mathscr{B} \to \mathscr{B}$. The spectrum

$\sigma(L)$ is subdivided into mutually disjoint subsets: the point spectrum, $\sigma_p(L)$, those values for which $L(\lambda)$ has a nontrivial null space; the continuous spectrum, $\sigma_c(L)$, those values λ for which $L(\lambda)$ is one-to-one but $L^{-1}(\lambda)$ is not bounded; the residual spectrum, those values λ for which $L(\lambda)$ has a bounded inverse but which is not in $\mathscr{R}$.

THEOREM 4.1. *Suppose that there is a root $Z \in \mathscr{R}$ such that*

$$L(Z) = \sum_{i=0}^{\infty} A_i Z^i = 0; \tag{4.3}$$

then:

(1) *there is an expression*

$$Q(\lambda) = \sum_{i=0}^{\infty} Q_i \lambda^i, Q_i = \sum_{j=0}^{\infty} A_{i+1+j} Z^j \tag{4.4}$$

such that the corresponding function

$$q(\lambda) = \sum_{i=0}^{\infty} \| Q_i \| \lambda^i \tag{4.5}$$

is entire and the identity

$$L(\lambda) = Q(\lambda)\,(\lambda I - Z) \tag{4.6}$$

is valid for all complex numbers λ;

(2) *the continuous spectrum of Z, $\sigma_c(Z) \subset \sigma_c(L)$*;

(3) *the point spectrum of Z, $\sigma_p(Z) \subset \sigma_p(L)$*;

(4) *the eigenvectors of Z are eigenvectors of L*;

(5) *the residual spectrum of Z not contained in $\sigma(Q)$ is contained in $\sigma(L)$.*

Proof. To see that $q(\lambda)$ is entire let λ be given and choose ρ larger than each of the numbers $\| Z \|$, $| \lambda |$, and one. Then

$$| q(\lambda) | \leqslant \sum_{i=0}^{\infty} \left[\sum_{j=0}^{\infty} \| A_{i+1+j} \| \rho^j \right] | \lambda |^i \leqslant \rho^{-1} \sum_{i=0}^{\infty} \frac{g^{(i+1)}(\rho)}{(i+1)!} \leqslant \rho^{-1} g(\rho) < \infty.$$

The identity (4.6) is obtained from expanding the right side:

$$Q(\lambda)(\lambda I - Z) = \sum_{i=0}^{\infty} \lambda^{i+1} Q_i - \sum_{i=0}^{\infty} \lambda^i Q_i Z = -Q_0 Z + \sum_{i=1}^{\infty} \lambda^i (Q_{i-1} - Q_i Z).$$

But $-Q_0 Z = A_0$ and

$$Q_{i-1} - Q_i Z = \sum_{j=0}^{\infty} A_{i+j} Z^i - \sum_{j=1}^{\infty} A_{i+j} Z^j = A_i , \qquad i \geqslant 1.$$

For (2), let $\{x_n\}$ be a sequence of approximate eigenvectors for a value λ_0 in $\sigma_c(Z)$, i.e.,

$$(\lambda_0 I - Z) x_n = \delta_n , \qquad \| x_n \| = 1, \qquad \| \delta_n \| \to 0 \qquad \text{as} \quad n \to \infty. \tag{4.7}$$

Then

$$\| L(\lambda_0) x_n \| \leqslant \| Q(\lambda_0) \| \, \| \delta_n \| \to 0 \qquad \text{as} \quad n \to \infty. \tag{4.8}$$

This argument applies also for (3) and (4) taking $x_n = x_0$, $\delta_n = 0$, $n = 0, 1, \ldots$. In effect, we obtain that (approximate) eigenvectors for Z belonging to λ_0 are (approximate) eigenvectors for L.

Finally, suppose $\lambda_0 \in \sigma_r(Z)$ and suppose on the contrary that $\lambda_0 \notin \sigma(L) \cup \sigma(Q)$.

Consider now expressions of the form

$$L(\lambda) = \lambda I - \sum_{i=0}^{\infty} \lambda^i A_i , \tag{4.9}$$

where the function

$$f(\lambda) = \sum_{i=0}^{\infty} \| A_i \| \lambda^i \tag{4.10}$$

satisfies the hypothesis of Theorem 1.1 and let Z

$$| Z | \leqslant a < b \tag{4.11}$$

be the resulting root of

$$L(Z) = Z - \sum_{i=0}^{\infty} A_i Z^i = 0 \tag{4.12}$$

Proposition 4.2. *Every complex number λ such that $a < | \lambda | \leqslant b$ is not in the spectrum of L.*

Proof. For $a < | \lambda | \leqslant b$, $\lambda^{-1} L(\lambda) = I - H(\lambda)$ where $\| H(\lambda) \| = | \lambda |^{-1} f(| \lambda |) < 1$.

In order to obtain approximations of $\sigma(L)$ we need the following extension of Rouche's Theorem.

LEMMA 4.3. *Let X be an operator in $\mathscr{R}$ and let D be a bounded Cauchy domain such that its boundary ∂D does not interesect $\sigma(X)$. Let $h(z)$ be analytic on D and on its boundary and let ν be the maximum modulus of $h(z)$ on ∂D. Denote by γ the maximum of $\| R(\lambda, X)\|$ where*

$$R(\lambda, X) = (\lambda I - X)^{-1} \tag{4.13}$$

genetically denotes the resolvent of X. Suppose $Y \in \mathscr{R}$ such that

$$\delta = \gamma \| X - Y \| < 1; \tag{4.14}$$

then:

(1) *∂D does not intersect $\sigma(Y)$ and*

$$\left\| \frac{1}{2\pi i} \int_{\partial D} h(z) \left(R(\lambda, X) - R(\lambda, Y)\right) d\lambda \right\| \leqslant \frac{|\partial D|}{2\pi} \frac{\nu\gamma\delta}{1-\delta}; \tag{4.15}$$

(2) *If*

$$|\partial D| \gamma\delta < 2\pi (1 - \delta) \tag{4.16}$$

then $\sigma(Y)$ intersects D if and only if $\sigma(X)$ intersect D;

(3) *$D \supset \sigma(X)$ if and only if $D \supset \sigma(Y)$ and if this is true*

$$\| h(X) - h(Y)\| \leqslant \frac{|\partial D|}{2\pi} \frac{\nu\gamma\delta}{1-\delta}. \tag{4.17}$$

Proof. For (1), express $\lambda I - Y = (I - X)\ R(\lambda, X))\ (\lambda I - X)$. Then because $\|(Y - X)\ R(\lambda, X)\| \leqslant \delta < 1$, $\lambda I - Y$ has an inverse

$$R(\lambda, Y) = R(\lambda, X) \sum_{i=0}^{\infty} [(Y - X) R(\lambda, X)]^i, \quad \lambda \in \partial D, \tag{4.18}$$

and the estimate (4.18) follows from the inequality

$$\| R(\lambda, Y) - R(\lambda, X)\| \leqslant \gamma \sum_{i=1}^{\infty} \delta^i = \frac{\gamma\delta}{1-\delta}, \quad \lambda \in \partial D, \tag{4.19}$$

and the maximum modulus principle.

For (2) we appeal to the well-known fact (cf. [83, p. 421]) that an operator X has spectrum in a Cauchy domain D if and only if the corresponding projection

$$P(X, D) = \frac{1}{2\pi i} \int_{\partial D} R(\lambda, X)\, d\lambda \tag{4.20}$$

is nonzero. Since nonzero projection operators have norm $\geqslant 1$, and since, from (4.15) and (4.16),

$$\| P(X, D) - P(Y, D)\| < 1 \tag{4.21}$$

$P(X, D)$ is nonzero if and only if $P(Y, D)$ is nonzero.

For (3), observe that because a projection operator P is idempotent ($P^2 = P$), $P = I$ if and only if P has an inverse in $\mathscr{R}$. In view of (4.19), $P(X, D) = I$ if and only if $P(Y, D) = I$ but $P(X, D)$ ($P(Y, D)$) is the identity if and only if $D \supset \sigma(X)$ ($D \supset \sigma(Y)$). Thus $D \supset \sigma(X)$ if and only if $D \supset \sigma(Y)$.

The inequality (4.17) now follows as a consequence of (4.16) and the Dunford-Taylor integral representation of a function h of an operator X (cf. [83] p. 431),

$$h(X) = \frac{1}{2\pi i} \int_{\partial D} h(\lambda)\, R(\lambda, X)\, d\lambda, \qquad D \supset \sigma(X). \tag{4.22}$$

THEOREM 4.4. *Let A_0, A_1,... be a sequence in $\mathscr{R}$ such that the function* (4.11) *satisfies the condition $f(b) < b$ in Theorem* 2.1 *and let Z_0, Z_1 be a sequence of iterates satisfying* (2.3). *Suppose that there is an integer n and a Cauchy domain D such that its boundary ∂D does not intersect $\sigma(Z_n)$. Let γ denote the maximum of $\| R(\lambda, Z_n)\|$ on ∂D and suppose further that n is chosen sufficiently large so that $\delta = (f'(\alpha))^n [\| Z_1 - Z_0 \|/(1 - f'(\alpha))]$ satisfies condition* (4.17) *then*:

(1) *if Z_n has spectrum in D then Z_k, $n \leqslant k \leqslant \infty$, has spectrum in D*;

(2) *if A_0 is compact and Z_n has spectrum in D then L has spectrum in D; L has point spectrum in D if D does not contain the origin*;

(3) *if $D \supset \sigma(Z_n)$ then $D \supset \sigma(Z_k)$, $n \leqslant k \leqslant \infty$, and for any function $h(Z)$ analytic on D and its boundary*

$$\| h(Z_n) - h(Z_k)\| \leqslant \frac{|\partial D|}{2\pi} \frac{\nu\gamma\delta}{1 - \delta}, \qquad n \leqslant k \leqslant \infty \tag{4.23}$$

Theorem 4.4 is obtained from combining Theorem 2.1, Theorem 4.1, Theorem 4.3, and the Fredholm alternative.

5. QUADRATIC EIGENVALUE PROBLEM

We now consider the special case of (4.2)

$$L(\lambda) = \lambda^2 A + \lambda B + C. \tag{5.1}$$

PROPOSITION 5.1. *Let Z_1, Z_2 be a complete pair of roots in $\mathscr{R}$ of*

$$L(Z) = AZ^2 + BZ + C = 0; \tag{5.2}$$

then for all complex numbers λ

$$L(\lambda) = A(Z_1 - Z_2)(\lambda I - Z_2)(Z_1 - Z_2)^{-1}(\lambda I - Z_1). \tag{5.3}$$

Proof. We begin with the identity (4.7) which reduces to, in this case,

$$L(\lambda) = (\lambda A + AZ_1 + B)(\lambda I - Z_1). \tag{5.4}$$

Assuming for the moment that $\lambda \notin \sigma(Z_1)$ and noting that $BZ_1 - BZ_2 = -(AZ_1{}^2 + C) + (AZ_2{}^2 + C) = A(Z_2{}^2 - Z_1{}^2)$ we obtain from (5.4) that $L(\lambda)\ (\lambda I - Z_1)^{-1}\ (Z_1 - Z_2) = A(Z_1 - Z_2)\ (\lambda I - Z_2)$. The identity (5.3), already shown to be valid for $\lambda \notin \phi(Z_1)$, holds for all complex numbers λ by the principle of analytic continuation.

As a corollary of Proposition 5.1 and Proposition 4.2 we have the following.

PROPOSITION 5.2. *If L has a pair of complete roots in $\mathscr{R}$, Z_1, Z_2, and if $\sigma(Z_1) \cup \sigma(Z_2) \cup \sigma(L)$ is not the entire complex plane then A has an inverse in R. In particular, if*

$$4 \| B^{-1}A \| \| B^{-1}C \| < 1, \tag{5.5}$$

$\| Z_1 \| \leqslant a$, $\| Z_2^{-1} \| \leqslant b_*^{-1}$, *as in Section* 3, *then A has an inverse in $\mathscr{R}$.*

For the sake of comparison we state the following Theorem of Krein and Langer [42] (*cf. Theorem* 7.1).

THEOREM 5.3. *Let B and C be bounded, positive, self-adjoint operators in a Hilbert space $(\mathscr{H}, (\cdot, \cdot))$. Let C be compact and let the condition*

$$4(Cx, x)(x, x) < (Bx, x)^2, \qquad 0 \neq x \in \mathscr{H}, \tag{5.6}$$

be satisfied (*this forces B to have a bounded inverse*). *Then*:

(1) *the expression*

$$L(\lambda) = \lambda^2 I + \lambda B + C \tag{5.7}$$

has only one root Z_j with the property that $Z_1{}^* Z_1 \leqslant C$;

(2) *the root Z_1 and the concomitant root $Z_2 = -B - Z_1{}^*$ are symmetric for one* (*and only one*) *uniformly positive operator $S = B + Z_1 - Z_1{}^* = -(B + Z_2 + Z_2{}^*) = Z_1 - Z_2$;*

(3) *the root* Z_1 *is similar to a negative compact operator whose spectrum lies in an interval* $[-\alpha_<, 0]$ *and whose eigenvectors (eigenvalues) exhaust all eigenvectors (eigenvalues) of the first kind of the expression*;

(4) *the root* Z_2 *is similar to a negative bounded operator whose spectrum lies in the interval* $[-\| B \| - \| C \|, -\alpha_>]$ $(\alpha_> > \alpha_<)$ *and whose eigenvectors (eigenvalues) exhaust all eigenvectors (eigenvalues) of the second kind of the expression* L;

(5) *the spectrum* $\sigma(L) = \sigma(Z_1) \cup \sigma(Z_2)$;

(6) *the roots* Z_1 *and* Z_2 *form a complete pair.*

Using the identity (5.3), the inequalities in (3.3), (3.6), and (3.9), Theorem 2.1 and Theorem 4.1, we state the following Banach space version of Theorem 5.3.

THEOREM 5.4. *Let* A, B, C, A^{-1}, B^{-1} *be bounded linear operators defined on a Banach space* $(\mathscr{B}, \| \cdot \|)$. *Let condition* (5.5) *be satisfied and let* $a < b$ *have their meaning as in Section* 3; *then*:

(1) *the expression* (5.1) *has one and only one root* Z_1 *of norm* $\leqslant b$ *and* $\| Z_1 \| \leqslant a$;

(2) *the concomitant root* Z_2 *is given by* (3.5);

(3) *the spectrum of* Z_1 *is contained in the disk* $| z | \leqslant a$ *and the entire spectrum (continuous spectrum, point spectrum, eigenvectors) of* Z_1 *exhausts the entire spectrum, continuous spectrum, point spectrum, eigenvectors) of* L *in* $[0, a]$ (Z_1 *is compact if* C *is compact*);

(4) *the spectrum of* Z_2 *is contained in the annulus*

$$b_* \leqslant | z | \leqslant | A^{-1}B | (1 + \| B^{-1}A \| a)$$

and the entire spectrum (continuous spectrum, point spectrum, eigenvectors) of Z_2 *exhausts the entire spectrum (continuous spectrum, point spectrum, eigenvectors) of* L *in* $b_* \leqslant | z | \leqslant | A^{-1}B | (1 + \| B^{-1}A \| a)$;

(5) *the spectrum* $\sigma(L) = \sigma(Z_1) \cup \sigma(Z_2)$;

(6) *the roots* Z_1 *and* Z_2 *form a complete pair.*

We point out that both roots in Theorem 5.4 are obtained via linear iteration and hence the entire spectrum of L is subject to the approximation technique used in Theorem 4.4. Note that if C is compact, Z_1 is compact and Z_2 is the sum of an invertable operator and a compact operator.

For $\lambda \notin \sigma(L)$ all the terms in (5.3) have inverses and

$$\begin{aligned} L^{-1}(\lambda) &= (\lambda I - Z_1)^{-1} (Z_1 - Z_2)(\lambda I - Z_2)^{-1} (Z_1 - Z_2)^{-1} A^{-1} \\ &= (\lambda I - Z_1)^{-1} [(\lambda I - Z_2) - (\lambda I - Z_1)] (\lambda I - Z_2)^{-1} (Z_1 - Z_2)^{-1} A^{-1} \\ &= [(\lambda I - Z_1)^{-1} - (\lambda I - Z_2)^{-1}] (Z_1 - Z_2)^{-1} A^{-1}. \end{aligned}$$

Thus,

$$(\lambda I - Z_1)^{-1} - (\lambda I - Z_2)^{-1} = L^{-1}(\lambda) A(Z_1 - Z_2). \tag{5.8}$$

Integrating both sides of (5.8) along the boundary ∂D of a Cauchy domain $D \supset \sigma(L)$ we obtain (cf. (4.20)) the following

PROPOSITION 5.5. *Let $h(\lambda)$ be analytic in a Cauchy domain $D \supset \sigma(L)$ then*

$$h(Z_1) - h(Z_2) = \left(\frac{1}{2\pi i} \int_{\partial D} h(\lambda) L^{-1}(\lambda) \, d\lambda \right) A(Z_1 - Z_2). \tag{5.9}$$

6. DISCUSSION

Operator equations and eigenparameter problems are used to solve linear systems. The type of conditions we have imposed are satisfied for stiff equations, singular perturbations and differential equations with small deviating arguments. We intend to deal with these special areas in forthcoming papers.

In the sections on eigenparameters we have assumed that the operators in question are bounded. This requirement is not essential since many of the proofs are algebraic in nature. Moreover, we may still apply the results of Sections 2 and 3 to the unbounded operator case by introducing a stronger topology (cf. the example cited in the beginning of Section 2).

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We express our gratitude to Dr. E. H. Rogers for his help in compiling the references on nonlinear eigenparameter problems. A brief scan of the paper mentioned will reveal that many of the authors were not familiar with the other work which had been done in the area. We hope that this present compilation, although not complete, will serve to inform prospective researchers.

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